Annali del Dipartimento di Filosofia (Nuova Serie), XVII (2011), pp. 135-166

Infinitary Modal Logic and Generalized Kripke Semantics

Pierluigi Minari

This paper deals with the *infinitary* modal propositional logic $\mathbf{K}_{\omega_1}$, featuring countable disjunctions and conjunctions. It is known that the *natural* infinitary extension $\mathbf{LK}^{\Box}_{\omega_1}$ (here presented as a *Tait-style* calculus, $\mathbf{TK}^{\sharp}_{\omega_1}$) of the standard sequent calculus $\mathbf{LK}^{\Box}_{p}$ for the propositional modal logic $\mathbf{K}$ is *incomplete* w.r.t. to Kripke semantics. It is also known that in order to axiomatize $\mathbf{K}_{\omega_1}$ one has to add to $\mathbf{LK}^{\Box}_{\omega_1}$ new initial sequents corresponding to the infinitary propositional counterpart BF_{ω_1} of the Barcan-formula. We introduce a generalization of Kripke semantics, and prove that $\mathbf{TK}^{\sharp}_{\omega_1}$ is sound and complete w.r.t. to this *generalized* semantics. By the same proof strategy, we show that the stronger system $\mathbf{TK}_{\omega_1}$, allowing *countably infinite* sequents, axiomatizes $\mathbf{K}_{\omega_1}$, although it provably doesn't admit cut-elimination.

Mathematics Subject Classification: 03B45, 03F05.

Keywords: *modal logic*, *infinitary logic*, *Kripke semantics*, *sequent-style calculi*, *cut-elimination*.

1. *Introduction*

Let $\mathbf{LK}_p$ be the propositional fragment of Gentzen's sequent calculus for classical logic. As is well-known (more or less since the mid 1950's, see e.g. [5]), the sequent calculus $\mathbf{LK}^{\Box}_p$ obtained by adding to $\mathbf{LK}_p$ the inference rule

$$\frac{\Gamma \Rightarrow \varphi}{\Box\Gamma \Rightarrow \Box\varphi}\,\mathsf{K} \quad (\text{where } \Box\Gamma := \{\Box\psi \mid \psi \in \Gamma\})$$

http://www.fupress.com/adf
ISSN 0394-5073 (print) ISSN 1824-3770 (online)

provides an adequate axiomatization of the minimal normal propositional modal logic $\mathbf{K}$, semantically defined as the set of all modal formulas which are valid in every Kripke frame. Furthermore, this axiomatization satisfies the subformula property, since it is not difficult to verify that $\mathbf{LK}_p^{\Box}$ admits cut-elimination. Of course, the same does hold for the corresponding multimodal system $\mathbf{LK}_p^{\Box_{\mathsf{A}}}$ based on an arbitrarily fixed set A of agents — below, we will identify without limitations A with the set of natural numbers ω.

Let us now move on to consider the *infinitary* (multi)modal version $\mathbf{K}_{\omega_1}$ of $\mathbf{K}$. In its language, featuring the infinitary connectives $\bigvee$ (*countable* disjunction) and $\bigwedge$ (*countable* conjunction) in place of the corresponding finitary connectives $\vee$ and $\wedge$, many interesting infinitary modal operators (typically fix-points operators) become directly *definable* — for instance that of *common knowledge*, C:

$$\mathsf{C}\varphi := \bigwedge\{\mathsf{E}^n\varphi \mid n \geq 1\}$$

where $\mathsf{E}\varphi$ (*everyone knows, that* φ) is defined as

$$\mathsf{E}\varphi := \bigwedge\{\Box_i\varphi \mid i < \omega\}$$

and

$$\mathsf{E}^n\varphi := \overbrace{\mathsf{E}\dots\mathsf{E}}^{n}\,\varphi$$

It is however a known fact (see e.g. [6], [8], [10]) that the *natural* infinitary extension $\mathbf{LK}_{\omega_1}^{\Box}$ (or $\mathbf{LK}_{\omega_1}^{\Box_\omega}$, in the multimodal version) of the sequent calculus $\mathbf{LK}_p^{\Box}$ ($\mathbf{LK}_p^{\Box_\omega}$), which is simply obtained by replacing the rules for $\vee$ and $\wedge$ with their infinitary counterparts

$$\frac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \bigvee\Phi}\,(\varphi \in \Phi) \qquad \frac{\dots\ \varphi, \Gamma \Rightarrow \Delta\ \dots\ (\mathit{all}\ \varphi \in \Phi)}{\bigvee\Phi, \Gamma \Rightarrow \Delta}$$

and

$$\frac{\dots\ \Gamma \Rightarrow \Delta, \varphi\ \dots\ (\mathit{all}\ \varphi \in \Phi)}{\Gamma \Rightarrow \Delta, \bigwedge\Phi} \qquad \frac{\varphi, \Gamma \Rightarrow \Delta}{\bigwedge\Phi, \Gamma \Rightarrow \Delta}\,(\varphi \in \Phi)$$

is not Kripke complete. In particular, the schema

$$(BF_{\omega_1}) \qquad \bigwedge \Box\Phi \to \Box \bigwedge \Phi$$

that is the infinitary propositional counterpart of the famous *Barcan-formula* of quantified modal logic, is trivially *valid in all Kripke frames*, but *is not derivable* in $\mathbf{LK}^{\Box}_{\omega_1}$.

Notice that, as a consequence, the basic modal logic of common knowledge **KC** (see e.g. [1]) cannot be embedded in $\mathbf{LK}^{\Box}_{\omega_1}$, since BF_{ω_1} (together with its *converse* CBF_{ω_1}, which is instead derivable in $\mathbf{LK}^{\Box}_{\omega_1}$), is essentially needed in order to derive the *fixed point axiom* $\mathsf{C}\varphi \leftrightarrow \mathsf{E}\varphi \wedge \mathsf{EC}\varphi$ of **KC**:

$$\begin{array}{cccc}
\mathsf{C}\varphi & \leftrightarrow & \mathsf{E}^1\varphi \wedge \mathsf{E}^2\varphi \wedge \mathsf{E}^3\varphi \wedge \ldots & \\
 & & \updownarrow & \\
 & & \mathsf{E}\varphi \wedge (\mathsf{E}(\mathsf{E}^1\varphi) \wedge \mathsf{E}(\mathsf{E}^2\varphi) \wedge \ldots) & \\
 & & \downarrow_{BF_{\omega_1}} \quad \uparrow_{CBF_{\omega_1}} & \\
 & & \mathsf{E}\varphi \wedge \mathsf{E}(\mathsf{E}^1\varphi \wedge \mathsf{E}^2\varphi \wedge \ldots) \quad \leftrightarrow & \mathsf{E}\varphi \wedge \mathsf{EC}\varphi
\end{array}$$

Indeed, BF_{ω_1} plays a key role in the axiomatization of $\mathbf{K}_{\omega_1}$. It has been proved by Y. Tanaka ([8]; see also [10]) that the sequent calculus $\mathbf{LK}^{\Box}_{\omega_1} \oplus BF_{\omega_1}$ — that is, $\mathbf{LK}^{\Box}_{\omega_1}$ plus all instances of $\Rightarrow BF_{\omega_1}$ as further initial sequents — axiomatizes $\mathbf{K}_{\omega_1}$. A Hilbert-style calculus ($\mathbf{KL}_{\omega_1}$) axiomatizing $\mathbf{K}_{\omega_1}$ and featuring BF_{ω_1} as an axiom had been earlier provided by S. Radev in [6].

An alternative route is followed in the present paper. We work with *Tait-style* (i.e. one-sided) sequent calculi — but this is of course not essential — and we hide BF_{ω_1}, so to speak, in the syntax. More precisely, after some preliminaries on the peculiar *NNF*-language we adopt (sect. 2), we consider (sect. 3) the two calculi $\mathbf{TK}^{\sharp}_{\omega_1}$ and $\mathbf{TK}_{\omega_1}$: the essential difference between them is that *sequents* are *finite* (sets of formulas) in $\mathbf{TK}^{\sharp}_{\omega_1}$, whereas they can also be *countably infinite* in $\mathbf{TK}_{\omega_1}$. It is a known fact that this difference doesn't matter as far as non modal infinitary logic is concerned; it does however matter when modal operators are added: while $\mathbf{TK}^{\sharp}_{\omega_1}$ is indeed nothing but the one-sided version of $\mathbf{LK}^{\Box}_{\omega_1}$, the calculus $\mathbf{TK}_{\omega_1}$ turns out to be equivalent to $\mathbf{LK}^{\Box}_{\omega_1} \oplus BF_{\omega_1}$.

As we said, $\mathbf{LK}^{\Box}_{\omega_1}$ (our $\mathbf{TK}^{\sharp}_{\omega_1}$) is *incomplete* w.r. to Kripke semantics. Yet it is a natural calculus to consider. In sect. 4 we introduce a *generalized* Kripke semantics (*standard* Kripke semantics being but a limit case of the generalized one) and show that, on the one side, $\mathbf{TK}^{\sharp}_{\omega_1}$ is valid w.r. to the generalized semantics while, on the other side, BF_{ω_1} admits a very simple *generalized* countermodel.

In sect. 5 we prove a *completeness* theorem for $\mathbf{TK}^{\sharp}_{\omega_1}$ w.r. to the *generalized* Kripke semantics, thus providing an adequate relational semantics for this system. Actually, one and the same proof strategy — a suitable adaptation of the familiar canonical model construction — gives also, as a bonus, a smooth completeness proof for $\mathbf{TK}_{\omega_1}$ w.r. to *standard* Kripke semantics, which is alternative to the completeness proofs given in [8], [10] for $\mathbf{LK}^{\Box}_{\omega_1} \oplus BF_{\omega_1}$, and in [6] for $\mathbf{KL}_{\omega_1}$.

The question concerning *cut-free axiomatizability* of $\mathbf{K}_{\omega_1}$ is discussed in the concluding sect. 6. In particular, we show by suitable counterexamples that $\mathbf{TK}_{\omega_1}$, as well as some natural variants of this calculus, *do not* admit cut-elimination. As far as we know, only one cut-free axiomatization of $\mathbf{K}_{\omega_1}$ is presently available, Tanaka's calculus $\mathbf{TLM}_{\omega_1}$ ([9]), whose sequents are *trees of standard sequents.* The cut-elimination theorem for $\mathbf{TLM}_{\omega_1}$ is however proved only semantically.

2. *The infinitary modal language* $\mathcal{L}^{\Box}_{\omega_1}$

The alphabet of our infinitary multimodal propositional language $\mathcal{L}^{\Box}_{\omega_1}$ comprises the symbols:

- $p_0, \widetilde{p}_0, p_1, \widetilde{p}_1, p_2, \widetilde{p}_2 \ \ldots$: denumerably many propositional atoms and negated propositional atoms (*literals*);
- $\bigwedge, \bigvee, \Box_i, \widetilde{\Box}_i \ (i < \omega)$: logical operators.

We denote by Lit (Lit^+) the set of all (positive) literals.

The *formulas* of the language $\mathcal{L}^{\Box}_{\omega_1}$ are generated starting from the literals by applying as usual the modal operators $\Box_i$ and

$\widetilde{\Box}_i$ $(i < \omega)$, and by forming *countable* disjunctions ($\bigvee$) and conjunctions ($\bigwedge$).

More precisely, the set FM of all $\mathcal{L}^{\Box}_{\omega_1}$*-formulas* is the least fixed-point of the monotone operator F such that

$$F(X) = X \cup \{\Box_i x \mid x \in X, i < \omega\} \cup \{\widetilde{\Box}_i x \mid x \in X, i < \omega\}$$
$$\cup \{\bigwedge Z \mid Z \subseteq X, |Z| \leq \omega\} \cup \{\bigvee Z \mid Z \subseteq X, |Z| \leq \omega\}$$

Equivalently,

$$\mathsf{FM} = \bigcup_{\alpha<\omega_1} \mathsf{FM}^{\alpha}$$

where FM^{α} $(\alpha < \omega_1)$ is defined by transfinite induction as follows:

(i) $\mathsf{FM}^0 = \mathsf{Lit}$,

(ii) $\mathsf{FM}^{\beta+1} = F(\mathsf{FM}^{\beta})$,

(iii) $\mathsf{FM}^{\lambda} = \bigcup_{\beta<\lambda} \mathsf{FM}^{\beta}$ (where λ is a limit ordinal).

The notion of *subformula* of a formula is defined as usual.

Notational conventions 2.1. Henceforth, the lower-case Greek letters φ, ψ, χ, possibly with indices, are used as metavariables for formulas. Capital Greek letters $\Gamma, \Delta, \Phi, \Psi, \ldots$ will range over *countable subsets* of FM, while capital Roman letters $C, D, \ldots$ will range over *finite subsets* of FM.

Notice that the connective '$\neg$' is not contained in the alphabet of $\mathcal{L}^{\Box}_{\omega_1}$; officially, $\mathcal{L}^{\Box}_{\omega_1}$-formulas are always in *negation normal form*. It is however convenient to introduce *negation* in the metalanguage. The map

$$\varphi \in \mathsf{FM} \;\mapsto\; \neg\varphi \in \mathsf{FM}$$

is defined inductively in the natural way:

(i) $\neg p_k := \widetilde{p}_k, \quad \neg\widetilde{p}_k := p_k,$

(ii) $\neg\Box_i\psi := \widetilde{\Box}_i\psi, \quad \neg\widetilde{\Box}_i\psi := \Box_i\psi,$

(iii) $\neg\bigwedge\Psi := \bigvee\{\neg\psi \mid \psi \in \Psi\}, \quad \neg\bigvee\Psi := \bigwedge\{\neg\psi \mid \psi \in \Psi\}.$

Thus $\neg\neg\varphi$ and φ are syntactically identical (notation: $\neg\varphi \equiv \varphi$). Observe that φ need not be a subformula of $\neg\varphi$.

Notational conventions 2.2.

- $\Phi, \psi := \Phi \cup \{\psi\}$; $\quad \Phi, \Psi := \Phi \cup \Psi$; $\quad \dots$;
- $\neg\Phi := \{\neg\varphi \mid \varphi \in \Phi\}$;
- $\varphi \wedge \psi := \bigwedge\{\varphi, \psi\}$; $\quad \varphi \vee \psi := \bigvee\{\varphi, \psi\}$;
- $\varphi \to \psi := \neg\varphi \vee \psi$;
- $\top := \bigwedge \emptyset$; $\quad \bot := \bigvee \emptyset$;
- $\Box_i\Phi := \{\Box_i\varphi \mid \varphi \in \Phi\}$; $\quad \widetilde{\Box}_i\Phi := \{\widetilde{\Box}_i\varphi \mid \varphi \in \Phi\}$.

Note that $\neg\top \equiv \bot$ and $\neg\bot \equiv \top$.

We conclude the present section by introducing one further notion, which however will not be used until sect. 5.

Consider a *countable* set Γ of formulas: as FM is uncountable, there exists an ordinal $\alpha < \omega_1$ such that $\Gamma \subseteq \mathsf{FM}^\alpha$. This justifies the following

Definition 2.3 (Environment). Let Γ be a *countable* subset of FM:

(i) $\mathtt{rk}(\Gamma) :=$ the least *limit* ordinal $\lambda < \omega_1$ s.t. $\Gamma \subseteq \mathsf{FM}^\lambda$;

(ii) $\mathcal{E}[\Gamma] := \mathsf{FM}^{\mathtt{rk}(\Gamma)}$.

We call $\mathcal{E}[\Gamma]$ the *environment* of Γ.

Lemma 2.4. For every countable set Γ of formulas, its environment $\mathcal{E}[\Gamma]$ has the following properties:

(1) $\mathcal{E}[\Gamma]$ is countably infinite;

(2) $\mathcal{E}[\Gamma]$ is closed under subformulas and under negation;

(3) $\mathcal{E}[\Gamma]$ is closed under *finite* conjunctions and disjunctions, as well as under $\Box_i$ and $\widetilde{\Box}_i$ $(i < \omega)$;

(4) for each formula $\psi \in \mathcal{E}[\Gamma]$ and $\bigwedge \Phi \in \mathcal{E}[\Gamma]$, and for every $i < \omega$, the formula $\Box_i \bigwedge\{\psi \vee \varphi \mid \varphi \in \Phi\}$ belongs to $\mathcal{E}[\Gamma]$.

Proof. It is easily verified that, for any $\beta < \omega_1$, FM^β satisfies (1) – (2), and that FM^β also satisfies (3) – (4) whenever β is a limit ordinal. The conclusion follows by Definition 2.3. $\Box$

3. *Tait-style infinitary modal calculi*

As anticipated in the introductory section, we work with *Tait-style* sequent calculi. Along with the peculiar *NNF*-syntax of our language $\mathcal{L}^{\Box}_{\omega_1}$, this format allows a considerable economy and elegance in the presentation of the proof systems under investigation.

The sequents to be derived are therefore not the usual *two-sided* sequents, but rather *one-sided* sequents, that is *sets of formulas*, having a *disjuctive* reading in the intended interpretation[1].

A choice concerning the *cardinality of sequents* must however be taken right from the start: shall we confine to *finite* sequents only? Or shall we allow countably *infinite* sequents too?

As far as infinitary classical *truth-functional* logic is concerned, it makes no essential difference which of the two alternatives we adopt (see e.g. [2]). Indeed, let us start by considering three Tait-style *non modal* propositional calculi $\mathbf{T}^{\sharp}_{\omega_1}$, $\mathbf{T}^{*}_{\omega_1}$ and $\mathbf{T}_{\omega_1}$, whose inference rules[2] are shown in Figures 1, 2 and 3.

$\mathbf{T}^{\sharp}_{\omega_1}$ derives *finite* sets of formulas, whereas both $\mathbf{T}^{*}_{\omega_1}$ and $\mathbf{T}_{\omega_1}$ derive *countable*, possibly infinite sets of formulas (recall the Notational conventions 2.1). Modulo this difference $\mathbf{T}^{\sharp}_{\omega_1}$ and $\mathbf{T}^{*}_{\omega_1}$ have the "same" inference rules; on the other side, $\mathbf{T}^{*}_{\omega_1}$ and $\mathbf{T}_{\omega_1}$ differ only in the $\bigvee$-introduction rule[3]. Notice also

[1] In a classical environment, there is an obvious correspondence between two-sided and one-sided sequents. '$\Gamma \Rightarrow \Delta$' $\rightsquigarrow$ '$\neg\Gamma, \Delta$', and 'Γ' $\rightsquigarrow$ '$\neg\Gamma_1 \Rightarrow \Gamma_2$' for each partition (Γ_1, Γ_2) of Γ.

[2] Where formulas are of course intended to belong to the $\Box_i$- and $\widetilde{\Box}_i$-less fragment $\mathcal{L}_{\omega_1}$ of $\mathcal{L}^{\Box}_{\omega_1}$. **Caution**: in all the calculi under consideration in this paper sequents are *sets*, not *multisets*. This means that *contraction* is hidden in the logical inference rules: the principal formula of an inference may occur as a side formula in the premise(s); e.g.

$$\frac{C, \varphi \vee \psi, \psi}{C, \varphi \vee \psi} \quad \text{and} \quad \frac{\Gamma, \Phi, \bigvee \Phi}{\Gamma, \bigvee \Phi}$$

are instances of `OR`, respectively `OR`$^{+}$.

[3] Of course, because of the presence of the rule `W`, the rule `OR` is a derived rule in $\mathbf{T}_{\omega_1}$.

Figure 1. $\mathbf{T}^{\sharp}_{\omega_1}$

$$\frac{}{p, \neg p}\ \texttt{ID}\ \ (p \in \mathsf{Lit}^{+}) \qquad \frac{C}{C, D}\ \texttt{W} \qquad \frac{C, \varphi \qquad D, \neg\varphi}{C, D}\ \texttt{CUT}$$

$$\frac{C, \varphi}{C, \bigvee \Phi}\ \texttt{OR}\ \ (\varphi \in \Phi) \qquad \frac{\ldots\ C, \varphi\ \ldots\ (\mathit{all}\ \varphi \in \Phi)}{C, \bigwedge \Phi}\ \texttt{AND}$$

Figure 2. $\mathbf{T}^{*}_{\omega_1}$

$$\frac{}{p, \neg p}\ \texttt{ID}\ \ (p \in \mathsf{Lit}^{+}) \qquad \frac{\Gamma}{\Gamma, \Delta}\ \texttt{W} \qquad \frac{\Gamma, \varphi \qquad \Delta, \neg\varphi}{\Gamma, \Delta}\ \texttt{CUT}$$

$$\frac{\Gamma, \varphi}{\Gamma, \bigvee \Phi}\ \texttt{OR}\ \ (\varphi \in \Phi) \qquad \frac{\ldots\ \Gamma, \varphi\ \ldots\ (\mathit{all}\ \varphi \in \Phi)}{\Gamma, \bigwedge \Phi}\ \texttt{AND}$$

Figure 3. $\mathbf{T}_{\omega_1}$

$$\frac{}{p, \neg p}\ \texttt{ID}\ \ (p \in \mathsf{Lit}^{+}) \qquad \frac{\Gamma}{\Gamma, \Delta}\ \texttt{W} \qquad \frac{\Gamma, \varphi \qquad \Delta, \neg\varphi}{\Gamma, \Delta}\ \texttt{CUT}$$

$$\frac{\Gamma, \Phi}{\Gamma, \bigvee \Phi}\ \texttt{OR}^{+} \qquad \frac{\ldots\ \Gamma, \varphi\ \ldots\ (\mathit{all}\ \varphi \in \Phi)}{\Gamma, \bigwedge \Phi}\ \texttt{AND}$$

that the generalized cut-rule with countably many premises

$$\frac{\Gamma, \Phi \qquad \ldots\ \neg\varphi, \Delta\ \ldots\ (\mathit{all}\ \varphi \in \Phi)}{\Gamma, \Delta}\ \texttt{CUT}^{+}$$

is admissible (as a derived rule) in $\mathbf{T}_{\omega_1}$; actually $\mathbf{T}_{\omega_1}$ is equivalent to the calculus obtained from $\mathbf{T}^*_{\omega_1}$ by replacing the rules $\mathtt{OR}^+$ and $\mathtt{CUT}$ with the rules $\mathtt{OR}$ and $\mathtt{CUT}^+$, respectively.

It is immediately verified that

(1) for an arbitrary formula φ, the sequents

$$\varphi, \neg\varphi \qquad \text{and} \qquad \top$$

are cut-free derivable in any of these calculi, the latter through a vacuous application of the $\mathtt{AND}$ rule;

(2) for every C and Γ,

$$\mathbf{T}^\sharp_{\omega_1} \vdash C \;\Leftrightarrow\; \mathbf{T}^*_{\omega_1} \vdash C \quad \text{and} \quad \mathbf{T}^*_{\omega_1} \vdash \Gamma \;\Rightarrow\; \mathbf{T}_{\omega_1} \vdash \Gamma$$

In fact, also the second arrow in (2) above can be reversed and, in a sense to be specified, the three calculi are *equivalent*. Let us write

- '$\models_{\omega_1} \Gamma$' to mean that $\bigvee \Gamma$ is truth-functionally valid: for every valuation $\mathsf{v} : \mathsf{Lit}^+ \longrightarrow \{0, 1\}$ there is some $\varphi \in \Gamma$ such that $\mathsf{v}(\varphi) = 1$ (v being extended from positive literals to arbitrary $\mathcal{L}_{\omega_1}$-formulas in the natural way);
- '$\vdash_0 \Gamma$' to mean that Γ is *cut-free derivable.*

Then the relations between the three calculi, their soundness and semantic completeness, as well as the cut-elimination property for $\mathbf{T}^\sharp_{\omega_1}$, can be summarized as follows on the basis of known results.

Proposition 3.1. Let C be a finite set of $\mathcal{L}_{\omega_1}$-formulas, and $\{\Gamma_1, \ldots, \Gamma_n\}$ be a finite, possibly empty set of countable sets of $\mathcal{L}_{\omega_1}$-formulas. Then the following are equivalent:

(1) $\mathbf{T}^\sharp_{\omega_1} \vdash_0 \bigvee \Gamma_1, \ldots, \bigvee \Gamma_n, C$

(2) $\mathbf{T}^\sharp_{\omega_1} \vdash \bigvee \Gamma_1, \ldots, \bigvee \Gamma_n, C$

(3) $\mathbf{T}^*_{\omega_1} \vdash \Gamma_1, \ldots, \Gamma_n, C$

(4) $\mathbf{T}_{\omega_1} \vdash \Gamma_1, \ldots, \Gamma_n, C$

(5) $\models_{\omega_1} \Gamma_1, \ldots, \Gamma_n, C$

Proof. (1) ⇒ (2), (3) ⇒ (4), (4) ⇒ (5): trivial.
(2) ⇒ (3): straightforward, because $\mathbf{T}^*_{\omega_1} \vdash \neg \bigvee \Gamma, \Gamma$.
(5) ⇒ (1): completeness and cut elimination for $\mathbf{T}^\sharp_{\omega_1}$ are proved in [7]. □

Notice that the syntactic proof of cut-elimination for $\mathbf{T}^\sharp_{\omega_1}$ given in [7] can be easily adapted to $\mathbf{T}^*_{\omega_1}$ and $\mathbf{T}_{\omega_1}$ (see e.g. [3]; cp. also [4] and [2]).

Let us now extend the above calculi with modal inference rules, with the aim of capturing — by means of an adequate infinitary Tait-style calculus — the infinitary modal propositional logic determined by the class of all Kripke frames, i.e. the infinitary version $\mathbf{K}_{\omega_1}$ of the (multi)modal system $\mathbf{K}$. Taking into account the need for a preliminary choice concerning the cardinality of sequents, we shall consider on the basis of the previous investigation the three candidates $\mathbf{TK}^\sharp_{\omega_1}$, $\mathbf{TK}^*_{\omega_1}$ and $\mathbf{TK}_{\omega_1}$ shown in Figure 4.

Figure 4. $\mathbf{TK}^\sharp_{\omega_1}$, $\mathbf{TK}^*_{\omega_1}$ **and** $\mathbf{TK}_{\omega_1}$

$$\mathbf{TK}^\sharp_{\omega_1} := \mathbf{T}^\sharp_{\omega_1} + \frac{\neg C, \varphi}{\neg \Box_i C, \Box_i \varphi}\, \mathtt{K}^\sharp_i \quad (i < \omega)$$

$$\mathbf{TK}^*_{\omega_1} := \mathbf{T}^*_{\omega_1} + \frac{\neg \Gamma, \varphi}{\neg \Box_i \Gamma, \Box_i \varphi}\, \mathtt{K}_i \quad (i < \omega)$$

$$\mathbf{TK}_{\omega_1} := \mathbf{T}_{\omega_1} + \frac{\neg \Gamma, \varphi}{\neg \Box_i \Gamma, \Box_i \varphi}\, \mathtt{K}_i \quad (i < \omega)$$

The present scenario turns out to be radically different from the previous (non modal) one: now the alternative “finite vs countable sequents” does matter! Indeed, we have:

- **(a)** $\mathbf{TK}^\sharp_{\omega_1}$ admits cut-elimination, but is *incomplete* (though clearly sound) w.r. to Kripke semantics;

(b) $\mathbf{TK}^{*}_{\omega_1}$ too admits cut-elimination and is *incomplete* (although clearly sound) w.r. to Kripke semantics; but it is not equivalent to $\mathbf{TK}^{\sharp}_{\omega_1}$ in the sense in which $\mathbf{T}^{\sharp}_{\omega_1}$ and $\mathbf{T}^{*}_{\omega_1}$ are equivalent according to Proposition 3.1;

(c) $\mathbf{TK}_{\omega_1}$ is instead *sound and complete* w.r. to Kripke semantics; yet it provably doesn't admit cut-elimination.

As to point **(a)**, a syntactic proof of cut-elimination for $\mathbf{TK}^{\sharp}_{\omega_1}$ can be obtained by adapting Tait's proof of the cut-elimination theorem for $\mathbf{T}^{\sharp}_{\omega_1}$ ([7], see also [9]) — the same does hold for $\mathbf{TK}^{*}_{\omega_1}$. The semantical incompleteness of $\mathbf{TK}^{\sharp}_{\omega_1}$ follows in turn as a consequence of cut-elimination, because of the following easily verifiable facts:

Fact 3.2. The schema ("*Barcan Formula*")

$$(BF_{\omega_1}) \qquad \bigwedge \Box_i \Phi \to \Box_i \bigwedge \Phi, \quad \text{or, as a finite sequent,} \quad \neg \bigwedge \Box_i \Phi, \Box_i \bigwedge \Phi$$

is valid in every Kripke model[4].

Fact 3.3. The instance

$$\neg \bigwedge \{\Box_i p_n \mid n < \omega\}, \Box_i \bigwedge \{p_n \mid n < \omega\}$$

of BF_{ω_1} has no *cut-free* derivation in $\mathbf{TK}^{\sharp}_{\omega_1}$.

Notice that BF_{ω_1} is (cut-free) derivable in $\mathbf{TK}_{\omega_1}$ as follows:

$$\dfrac{\dfrac{\dfrac{\cdots \left\{ \dfrac{}{\neg\Phi, \varphi}\,\texttt{ID, W} \right\} \cdots \quad (\varphi \in \Phi)}{\neg\Phi, \bigwedge \Phi}\,\texttt{AND}}{\neg\Box_i\Phi, \Box_i \bigwedge \Phi}\,\texttt{K}_i}{\neg \bigwedge \Box_i\Phi, \Box_i \bigwedge \Phi}\,\texttt{OR}^{+}$$

On the other side, the "*converse Barcan Formula*"

$$\Box_i \bigwedge \Phi \to \bigwedge \Box_i \Phi$$

[4]See the next section.

is derivable already in $\mathbf{TK}^{\sharp}_{\omega_1}$:

$$\dfrac{\cdots\left\{\dfrac{\dfrac{\neg\varphi,\varphi}{\neg\bigwedge\Phi,\varphi}\,\texttt{OR}}{\neg\Box_i\bigwedge\Phi,\Box_i\varphi}\,\texttt{K}_i\right\}\cdots\;(\varphi\in\Phi)}{\neg\Box_i\bigwedge\Phi,\bigwedge\Box_i\Phi}\,\texttt{AND}$$

As to point **(b)**, it is immediate to verify that $\mathbf{TK}^{\sharp}_{\omega_1}$ and $\mathbf{TK}^{*}_{\omega_1}$ derive the same *finite* sequents,

$$(3.1)\qquad \mathbf{TK}^{\sharp}_{\omega_1}\vdash C \;\Leftrightarrow\; \mathbf{TK}^{*}_{\omega_1}\vdash C \qquad \text{for every } C,$$

like the corresponding non modal calculi. Hence BF_{ω_1} is not derivable in $\mathbf{TK}^{*}_{\omega_1}$ as well. On the other side, contrary to Proposition 3.1,

$$(3.2)\qquad \mathbf{TK}^{*}_{\omega_1}\vdash\Gamma \;\not\Rightarrow\; \mathbf{TK}^{\sharp}_{\omega_1}\vdash\bigvee\Gamma$$

For instance, let $\Psi := \{\neg\Box_i p_0, \neg\Box_i p_1, \neg\Box_i p_2 \ldots, \Box_i \bigwedge_{n<\omega} p_n\}$. Then $\mathbf{TK}^{*}_{\omega_1}\vdash\Psi$

$$\dfrac{\dfrac{\cdots\left\{\dfrac{}{\{\neg p_n\}_{n<\omega},p_m}\,\texttt{ID, W}\right\}\cdots\;(m<\omega)}{\{\neg p_n\}_{n<\omega},\bigwedge_{n<\omega}p_n}\,\texttt{AND}}{\{\neg\Box_i p_n\}_{n<\omega},\Box_i\bigwedge_{n<\omega}p_n}\,\texttt{K}_i$$

but, as a semantic argument in the next section (Fact 4.4) will show[5], $\mathbf{TK}^{\sharp}_{\omega_1}\nvdash\bigvee\Psi$.

As to the two claims made in point **(c)**, these will be addressed in section 5, where the semantic completeness of $\mathbf{TK}_{\omega_1}$ is proved, and in section 6, where the question concerning the non eliminability of the cut rule in this calculus is discussed. Notice that merely on the basis of what has been established so far, in particular (3.1) and (3.2) above, we already know that $\mathbf{TK}^{*}_{\omega_1}$ *is not closed* under the inference rule $\texttt{OR}^{+}$ characteristic of $\mathbf{TK}_{\omega_1}$.

[5]A simple syntactic argument is also at hand, by using Fact 3.3 and (see below) point (ii) in the proof of Proposition 3.4.

We conclude this section by stating a simple equivalence result, by which the key role played in the present context by the Barcan Formula BF_{ω_1} is made fully evident.

Let $\mathbf{TKB}^{\sharp}_{\omega_1}$ and $\mathbf{TKB}^{*}_{\omega_1}$ be the calculi obtained from $\mathbf{TK}^{\sharp}_{\omega_1}$, resp. $\mathbf{TK}^{*}_{\omega_1}$, by adding all the instances of BF_{ω_1} as new initial sequents. Then:

Proposition 3.4. For every countable set Γ of formulas, the following are equivalent:

(1) $\mathbf{TK}_{\omega_1} \vdash \Gamma$

(2) $\mathbf{TKB}^{*}_{\omega_1} \vdash \Gamma$

(3) $\mathbf{TKB}^{\sharp}_{\omega_1} \vdash \bigvee \Gamma$

Proof. (2) $\Rightarrow$ (1): obvious, since $\mathbf{TK}_{\omega_1} \vdash BF_{\omega_1}$.
(3) $\Rightarrow$ (2): it is sufficient to observe that the inversion of $\texttt{OR}^+$ is admissible (as a derived rule) in $\mathbf{TK(B)}^{*}_{\omega_1}$:

$$\dfrac{\Gamma, \bigvee \Phi \qquad \dfrac{\cdots \left\{ \dfrac{}{\neg\varphi, \Phi}\, \texttt{ID, W} \right\} \cdots \quad (\varphi \in \Phi)}{\neg \bigvee \Phi, \Phi}\, \texttt{AND}}{\Gamma, \Phi}\, \texttt{CUT}$$

(1) $\Rightarrow$ (3): first of all, observe that for every Γ, Δ, C :

(i) $\mathbf{TK(B)}^{\sharp}_{\omega_1} \vdash \neg \bigvee \Gamma, \bigvee (\Gamma \cup \Delta)$

(ii) $\mathbf{TK(B)}^{\sharp}_{\omega_1} \vdash \neg \bigvee (\Gamma \cup C), \bigvee \Gamma, C$

(iii) $\mathbf{TK(B)}^{\sharp}_{\omega_1} \vdash \neg \bigvee (\Gamma \cup \Delta), \bigvee \Gamma, \bigvee \Delta$

The easy verification is left to the reader.
Next, given a $\mathbf{TK}_{\omega_1}$-derivation $\mathfrak{D} \vdash \Gamma$, we produce a derivation $\mathfrak{D}' \vdash \bigvee \Gamma$ in $\mathbf{TKB}^{\sharp}_{\omega_1}$ arguing by transfinite induction on the height of $\mathtt{h}(\mathfrak{D}) < \omega_1$ of $\mathfrak{D}$ (which is defined in the natural way) and taking cases according to the final inference R of $\mathfrak{D}$. The case $R = \texttt{ID}$ is trivial; the cases $R = \texttt{W}, R = \texttt{CUT}$ are easily dealt with using the I.H. together with (i) and (ii) above. Let us spell out the details only for the cases $R = \texttt{OR}^+$ ($R = \texttt{AND}$ is similar) and $R = \texttt{K}_i$.
[$R = \texttt{OR}^+$]: then, taking into account the possibility that the

principal formula is a side formula in the premise, $\mathfrak{D}$ has the form:

$$\dfrac{\begin{array}{c}\vdots\\ \Delta, \bigvee\Phi, \Phi\end{array}}{\Delta, \bigvee\Phi}\ \texttt{OR}^{+}$$

We obtain $\mathfrak{D}' \vdash \bigvee(\Delta \cup \{\bigvee\Phi\})$ in $\mathbf{TKB}^{\sharp}_{\omega_1}$ as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\ \textbf{I.H.}\\ \bigvee(\Delta \cup \Phi \cup \{\bigvee\Phi\})\end{array}}{\bigvee(\Delta \cup \Phi), \bigvee\Phi}\ \texttt{CUT}\text{ with (ii)}}{\bigvee\Delta, \bigvee\Phi}\ \texttt{CUT}\text{ with (iii)}}{\bigvee(\Delta \cup \{\bigvee\Phi\}), \bigvee\Phi}\ \texttt{CUT}\text{ with (i)}}{\bigvee(\Delta \cup \{\bigvee\Phi\})}\ \texttt{OR}$$

$[\,R = \texttt{K}_i\,]$: then $\mathfrak{D}$ has the form:

$$\dfrac{\begin{array}{c}\vdots\\ \neg\Delta, \varphi\end{array}}{\neg\Box_i\Delta, \Box_i\varphi}\ \texttt{K}_i$$

We obtain $\mathfrak{D}' \vdash \bigvee(\neg\Box_i\Delta \cup \{\Box_i\varphi\})$ in $\mathbf{TKB}^{\sharp}_{\omega_1}$ as follows:

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}BF_{\omega_1}\\ \bigvee\neg\Box_i\Delta, \Box_i\bigwedge\Delta\end{array} \qquad \dfrac{\dfrac{\begin{array}{c}\vdots\ \textbf{I.H.}\\ \bigvee(\neg\Delta \cup \{\varphi\})\end{array}}{\neg\bigwedge\Delta, \varphi}\ \texttt{CUT}\text{ with (ii)}}{\neg\Box_i\bigwedge\Delta, \Box_i\varphi}\ \texttt{K}^{\sharp}_i}{\bigvee\neg\Box_i\Delta, \Box_i\varphi}\ \texttt{CUT}}{\bigvee(\neg\Box_i\Delta \cup \{\Box_i\varphi\}), \Box_i\varphi}\ \texttt{CUT}\text{ with (i)}}{\bigvee(\neg\Box_i\Delta \cup \{\Box_i\varphi\})}\ \texttt{OR}$$

$\Box$

4. *Generalized (and standard) Kripke frames*

As we saw, $\mathbf{TK}^{\sharp}_{\omega_1}$ is *incomplete* w.r. to Kripke semantics. In order to provide this calculus with a (possibly natural) *adequate* semantics, we introduce here a generalization of standard Kripke models.

A *generalized* (multimodal) Kripke frame is a pair

$$\mathsf{G} = \langle W, \{\mathfrak{R}_i\}_{i<\omega}\rangle$$

where W is a non empty set and, for each $i < \omega$, $\mathfrak{R}_i$ is a *non empty family* of binary relations over W which is *downward closed* w.r. to inclusion, i.e. it satisfies:

(DD) $$(\forall R \in \mathfrak{R}_i)(\forall S \in \mathfrak{R}_i)(\exists T \in \mathfrak{R}_i)(T \subseteq R \cap S)$$

Standard Kripke frames are of course a special case of generalized Kripke frames, namely the case in which the family $\mathfrak{R}_i$ reduces to a singleton $\{R_i\}$ — singletons trivially satisfy the condition (DD) — for each $i < \omega$.
When dealing with standard frames and models (see below) we will keep to the familiar way of presentation, by identifying $\{R_i\}$ with R_i. In other words, we write $\mathsf{S} = \langle W, \{R_i\}_{i<\omega}\rangle$ instead of $\mathsf{S} = \langle W, \{\{R_i\}\}_{i<\omega}\rangle$, when S is standard.

A valuation over a generalized frame G is, as usual, a map

$$\mathsf{v} : \mathsf{Lit}^+ \longrightarrow 2^W$$

Finally, a *generalized Kripke model* is a triple

$$\mathcal{M} = \langle W, \{\mathfrak{R}_i\}_{i<\omega}, \mathsf{v}\rangle$$

where $\mathsf{G} = \langle W, \{\mathfrak{R}_i\}_{i<\omega}\rangle$ is a generalized Kripke frame and v is a valuation over G.

Given a generalized model $\mathcal{M} = \langle W, \{\mathfrak{R}_i\}_{i<\omega}, \mathsf{v}\rangle$, an element w of $W_{\mathcal{M}} = W$ and a formula $\varphi \in \mathsf{FM}$, the relation

$$\mathcal{M}, w \vDash_g \varphi$$

is defined inductively as follows:

(i) $\mathcal{M}, w \vDash_g p_k$ iff $w \in \mathsf{v}(p_k)$

(ii) $\mathcal{M}, w \vDash_g \widetilde{p}_k$ iff $w \notin \mathsf{v}(p_k)$

(iii) $\mathcal{M}, w \vDash_g \bigwedge \Psi$ iff $\mathcal{M}, w \vDash_g \psi$ for each $\psi \in \Psi$

(iv) $\mathcal{M}, w \vDash_g \bigvee \Psi$ iff $\mathcal{M}, w \vDash_g \psi$ for some $\psi \in \Psi$

(v) $\mathcal{M}, w \vDash_g \Box_i\psi$ iff there exists a relation $R \in \mathfrak{R}_i$ such that, for every $u \in W$, wRu implies $\mathcal{M}, u \vDash_g \psi$ $(i < \omega)$

(vi) $\mathcal{M}, w \vDash_g \tilde{\Box}_i\psi$ iff for each $R \in \mathfrak{R}_i$ there is a state $u \in W$ such that wRu and $\mathcal{M}, u \nvDash_g \psi$ $(i < \omega)$

where '$\mathcal{M}, w \nvDash_g \varphi$' is short for '*not* $(\mathcal{M}, w \vDash_g \varphi)$'.

Notice that, for every formula φ, we have

$$\mathcal{M}, w \vDash_g \neg\varphi \quad \text{iff} \quad \mathcal{M}, w \nvDash_g \varphi$$

as expected.

Truth of a formula φ in a generalized model $\mathcal{M}$, in symbols

$$\mathcal{M} \models_g \varphi$$

as well as *generalized universal validity* of a formula, in symbols

$$\models_g \varphi$$

are defined as usual:

- $\mathcal{M} \models_g \varphi$ iff $\mathcal{M}, w \vDash_g \varphi$ for all $w \in W_{\mathcal{M}}$;
- $\models_g \varphi$ iff $\mathcal{M} \models_g \varphi$ for all generalized models $\mathcal{M}$.

All these notions are extended to arbitrary countable sets of formulas Γ according to the following

Notational conventions 4.1.

- $\mathcal{M}, w \vDash_g \Gamma$ iff $\mathcal{M}, w \vDash_g \varphi$ *for some* $\varphi \in \Gamma$;
- $\mathcal{M} \models_g \Gamma$ iff $\mathcal{M}, w \vDash_g \Gamma$ for all $w \in W_{\mathcal{M}}$;
- $\models_g \Gamma$ iff $\mathcal{M} \models_g \Gamma$ for all generalized models $\mathcal{M}$.

Caution: note the *disjunctive reading* of $\mathcal{M}, w \vDash_g \Gamma$!

Standard models now coincide with *generalized models based on standard frames* $\mathsf{S} = \langle W, \{R_i\}_{i<\omega}\rangle$. In fact, as it is immediately seen, in this case the two *non-standard* clauses (v) and (vi) of the above inductive definition become the usual:

(v)$_{st}$ $\mathcal{M}, w \vDash \Box_i\psi$ iff wR_iu implies $\mathcal{M}, u \vDash \psi$ for every $u \in W$ $(i < \omega)$

(vi)$_{st}$ $\mathcal{M}, w \vDash \tilde{\Box}_i\psi$ iff there exists a state $u \in W$ s.t. wR_iu and $\mathcal{M}, u \nvDash \psi$ $(i < \omega)$

and so the satisfaction relation '$\mathcal{M}, w \vDash_g \varphi$' boils down to the familiar one '$\mathcal{M}, w \vDash \varphi$'.
Henceforth, when dealing with *standard* models $\mathcal{M}$, we will drop the index g from '$\mathcal{M}, w \vDash_g \varphi$' and from all the related notions, including those of Convention 4.1.

Trivially, the three calculi $\mathbf{TK}^{\sharp}_{\omega_1}$, $\mathbf{TK}^{*}_{\omega_1}$ and $\mathbf{TK}_{\omega_1}$ are valid w.r. to the standard Kripke semantics. Yet, after having generalized the notions of Kripke frame, Kripke model and universal validity in the way described above, we can easily realize that the soundness of the characteristic *schema*

$$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$$

of the minimal normal modal system **K** *is not lost.* As a consequence, we have that the infinitary system $\mathbf{TK}^{\sharp}_{\omega_1}$ is in fact *valid* w.r. to the *generalized* semantics.

Theorem 4.2 (**g-Validity for** $\mathbf{TK}^{\sharp}_{\omega_1}$).
For every finite set C of $\mathcal{L}^{\Box}_{\omega_1}$-formulas:

$$\vdash^{\sharp} C \quad \Rightarrow \quad \models_g C$$

Proof. It is sufficient to check that the modal rules $\mathsf{K}^{\sharp}_i$ of $\mathbf{TK}^{\sharp}_{\omega_1}$ preserve truth in any generalized model $\mathcal{M} = \langle W, \{\mathfrak{R}_i\}_{i<\omega}, \mathsf{v}\rangle$; that is, for every $i < \omega$:

$$\mathcal{M} \models_g \{\neg\psi_1, \dots \neg\psi_n, \varphi\} \;\Rightarrow\; \mathcal{M} \models_g \{\neg\Box_i\psi_1, \dots \neg\Box_i\psi_n, \Box_i\varphi\}$$

So, assume that $\mathcal{M} \models_g \{\neg\psi_1, \dots \neg\psi_n, \varphi\}$ and suppose, towards a contradiction, that $\mathcal{M} \not\models_g \{\neg\Box_i\psi_1, \dots \neg\Box_i\psi_n, \Box_i\varphi\}$; then

$$\mathcal{M}, w \vDash_g \Box_i\psi_1,\ \dots\ , \mathcal{M}, w \vDash_g \Box_i\psi_n \tag{4.1}$$

and

$$\mathcal{M}, w \nvDash_g \Box_i\varphi \tag{4.2}$$

for some $w \in W$.
By (4.1), there exist relations $R_1, \dots, R_n \in \mathfrak{R}_i$ such that

$$\text{for } 1 \leq k \leq n: \quad (\forall u \in W)(wR_k u \Rightarrow \mathcal{M}, u \vDash_g \psi_k) \tag{4.3}$$

Applying the downward directedness condition (DD) to $\mathfrak{R}_i$, let $T \in \mathfrak{R}_i$ be such that $T \subseteq R_1 \cap \ldots \cap R_n$. Then, by (4.3):

$$\text{(4.4)} \qquad \text{for } 1 \leq k \leq n: \quad (\forall u \in W)(wTu \Rightarrow \mathcal{M}, u \vDash_g \psi_k)$$

On the other side, it follows by (4.2) and $T \in \mathfrak{R}_i$ that there exists a state $t \in W$ such that

$$\text{(4.5)} \qquad wTt \text{ and } \mathcal{M}, t \nvDash_g \varphi$$

Hence, by (4.4) and (4.5):

$$\text{(4.6)} \qquad \mathcal{M}, t \nvDash_g \{\neg\psi_1, \ldots, \neg\psi_n, \varphi\}$$

in contrast with our assumption $\mathcal{M} \models_g \{\neg\psi_1, \ldots \neg\psi_1, \varphi\}$. □

Now, we show that the Barcan-Formula BF_{ω_1} is not universally valid w.r. to the *generalized* Kripke semantics.

Fact 4.3 (g-countermodel for BF_{ω_1}). $\not\models_g BF_{\omega_1}$.

Proof. Let us consider the generalized Kripke model

$$\mathcal{C} = \langle \mathbb{N}^*, \{\mathfrak{R}_i\}_{i<\omega}, \mathsf{v} \rangle$$

where:

- $\mathbb{N}^* := \mathbb{N} \cup \{*\}$ (the natural numbers plus a new state $*$);
- for every $i < \omega$, $\mathfrak{R}_i = \mathfrak{R} := \{R_X \mid X \in \mathit{cof}(\mathbb{N})\}$, where:
  - $\mathit{cof}(\mathbb{N})$ is the set of all *cofinite* subsets of $\mathbb{N}$,
  - $R_X := \{\langle *, n \rangle \mid n \in X\}$;
- $\mathsf{v}(p_n) := \mathbb{N} \setminus \{n\}$, for each $p_n \in \mathsf{Lit}^+$.

The so defined family $\mathfrak{R}$ of relations satisfies the condition (DD) since on the one side cofinite subsets of $\mathbb{N}$ are closed under finite intersection, and on the other side, by definition

$$R_X \cap R_Y = R_{X \cap Y} \in \mathfrak{R} \quad (X, Y \in \mathit{cof}(\mathbb{N})).$$

Now, for $i < \omega$ we have, by construction, that for each $n \geq 0$:

$$\text{(4.7)} \qquad (\forall w \in \mathbb{N}^*)(*R_{\mathbb{N} \setminus \{n\}} w \;\Rightarrow\; \mathcal{C}, w \vDash_g p_n)$$

whence

$$\text{(4.8)} \qquad \mathcal{C}, * \vDash_g \Box_i p_n$$

Thus

$$(4.9) \qquad \mathcal{C}, * \vDash_g \bigwedge\{\Box_i p_n \mid n < \omega\}$$

On the other side, for every $X \in \mathit{cof}(\mathbb{N})$ and every $k \in X$:

$$(4.10) \qquad *R_X k \quad \mathit{and} \quad \mathcal{C}, k \nvDash_g p_k$$

Hence

$$(4.11) \qquad \mathcal{C}, * \nvDash_g \Box_i \bigwedge\{p_n \mid n < \omega\}$$

We conclude from (4.9) and (4.11) that

$$\mathcal{C} \not\models_g \bigwedge\{\Box_i p_n \mid n < \omega\} \to \Box_i \bigwedge\{p_n \mid n < \omega\}$$

the latter formula being an instance of BF_{ω_1}. $\Box$

Hence, Theorem 4.2 and Fact 4.3 supplement the underivability proof of BF_{ω_1} in $\mathbf{TK}^{\sharp}_{\omega_1}$, mentioned in the previous section, with a semantical argument. We can also use the above generalized Kripke model $\mathcal{C}$, together with Theorem 4.2, to see that the formula $\bigvee\{\neg\Box_i p_0, \neg\Box_i p_1, \neg\Box_i p_2 \ldots, \Box_i \bigwedge_{n<\omega} p_n\}$ is not derivable in $\mathbf{TK}^{\sharp}_{\omega_1}$, as claimed without proof in sect. 3. Indeed, by (4.8) and (4.2), we have

Fact 4.4. $\mathcal{C}, * \nvDash_g \bigvee\{\neg\Box_i p_0, \neg\Box_i p_1, \neg\Box_i p_2 \ldots, \Box_i \bigwedge_{n<\omega} p_n\}$.

In conclusion, observe that the results mentioned in point **(b)** of section 3 show that $\mathbf{TK}^{*}_{\omega_1}$ *is not* sound w.r. to the generalized Kripke semantics. Of course, both $\mathbf{TK}^{*}_{\omega_1}$ and $\mathbf{TK}_{\omega_1}$ are sound w.r. to the narrower class of all the generalized Kripke frames $\mathsf{G} = \langle W, \{\mathfrak{R}_i\}_{i<\omega}\rangle$ satisfying the *countable* downward directedness condition

$$(\mathrm{DD}_{\omega_1}) \quad (\forall \mathfrak{S} \subseteq \mathfrak{R}_i)(|\mathfrak{S}| \leq \omega \;\to\; (\exists T \in \mathfrak{R}_i)(\forall S \in \mathfrak{S})(T \subseteq S))$$

However, *only* $\mathbf{TK}_{\omega_1}$ is also *complete* w.r. to this special class of generalized frames. Thus the problem of finding an adequate Kripke-style semantic characterization of $\mathbf{TK}^{*}_{\omega_1}$ remains open.

5. *Completeness theorems for* $\mathbf{TK}^{\sharp}_{\omega_1}$ *and* $\mathbf{TK}_{\omega_1}$

We are going to prove, via a *canonical model* technique, a completeness theorem for $\mathbf{TK}^{\sharp}_{\omega_1}$ w.r. to the *generalized* Kripke semantics, and a completeness theorem for $\mathbf{TK}_{\omega_1}$ w.r. to the *standard* Kripke semantics. As the reader will see, our proofs of the two results run closely parallel and, in fact, eventually diverge only in one very specific key point.

We start with the common part of the two proofs. The notion of *environment* $\mathcal{E}[\Gamma]$ of a countable set Γ of formulas (see sect. 2) is employed here for the first time. In the following, we will often make a tacit use of Lemma 2.4.

By convenience, let us denote by $\mathbf{J}$ an arbitrarily fixed element of $\{\mathbf{TK}^{\sharp}_{\omega_1}, \mathbf{TK}_{\omega_1}\}$. Recall that the rule `OR` is available also in $\mathbf{TK}_{\omega_1}$ as a derived rule.

Definition 5.1. Let $\Gamma \subseteq \mathsf{FM}$:

(i) $\mathbf{J} \rhd \Gamma \ := \ \mathbf{J} \vdash C$ for some *finite* subset C of Γ;

(ii) Γ is $\mathbf{J}$-*consistent* iff $\mathbf{J} \not\rhd \neg\Gamma$;

(iii) Γ is $\mathbf{J}$-*saturated* iff:

(a) Γ is $\mathbf{J}$-*consistent*,

(b) $\Gamma \cup \neg\Gamma = \mathcal{E}[\Gamma]$,

(c) for all Φ such that $\bigwedge \Phi \in \mathcal{E}[\Gamma] : \Phi \subseteq \Gamma \Rightarrow \bigwedge \Phi \in \Gamma$;

(iv) Γ is $\mathbf{J}$-*uniform* iff for every Φ such that $\bigwedge \Phi \in \mathcal{E}[\Gamma]$: if $\mathbf{J} \rhd \neg\Gamma, \varphi$ for each $\varphi \in \Phi$, then $\mathbf{J} \rhd \neg\Gamma, \bigwedge \Phi$.

Lemma 5.2. [**Saturated sets**] Any $\mathbf{J}$-saturated set Γ satisfies:

(1) For all $\varphi \in \mathcal{E}[\Gamma]$, either $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$, and not both.

(2) For all $\varphi \in \mathcal{E}[\Gamma]$, if $\mathbf{J} \rhd \neg\Gamma, \varphi$ then $\varphi \in \Gamma$.

(3) For all Φ such that $\bigwedge \Phi \in \mathcal{E}[\Gamma]$: $\Phi \subseteq \Gamma$ iff $\bigwedge \Phi \in \Gamma$.

(4) For all Φ such that $\bigvee \Phi \in \mathcal{E}[\Gamma]$: $\Phi \cap \Gamma \neq \emptyset$ iff $\bigvee \Phi \in \Gamma$.

Proof.
(1): immediate, by (a) and (b) of Definition 5.1.(iii).
(2): suppose $\mathbf{J} \vdash \neg C, \varphi$ for some $C \subseteq \Gamma$. If $\varphi \notin \Gamma$ then, by (1), $\neg\varphi \in \Gamma$ and so $D := C \cup \{\neg\varphi\}$ is finite subset of Γ such that

$\mathbf{J} \vdash \neg D$, against the $\mathbf{J}$-consistency of Γ.
(3): if Φ is such that $\bigwedge \Phi \in \mathcal{E}[\Gamma]$, then also $\Phi \subseteq \mathcal{E}[\Gamma]$ by Definition 2.3 and Lemma 2.4. Suppose now that $\bigwedge \Phi \in \Gamma$, and let $\varphi \in \Phi$: as $\mathbf{J} \vdash \neg \bigwedge \Phi, \varphi$ trivially, we have that $\mathbf{J} \rhd \neg\Gamma, \varphi$ and so that $\varphi \in \Gamma$ by (2). Hence $\Phi \subseteq \Gamma$. The converse direction is given by (c) of Definition 5.1.(iii).
(4): by (1) and (3), $\bigvee \Phi \in \Gamma$ iff $\bigwedge \neg\Phi \notin \Gamma$ iff $\neg\Phi \not\subseteq \Gamma$ iff $\Phi \cap \Gamma \neq \emptyset$. $\square$

Fact 5.3 (Distributivity of $\vee$ over $\bigwedge$).

$$\mathbf{J} \vdash_{(0)} \psi \vee \bigwedge \Phi \leftrightarrow \bigwedge\{\psi \vee \varphi \mid \varphi \in \Phi\}$$

Proof.

$$\dfrac{\cdots\left\{\dfrac{\dfrac{\neg\psi, \psi, \varphi \qquad \neg\varphi, \psi, \varphi}{\neg(\psi \vee \varphi), \psi, \varphi}\,\texttt{AND}}{\neg \bigwedge\{\psi \vee \varphi \mid \varphi \in \Phi\}, \psi, \varphi}\,\texttt{OR}\right\}\cdots\ (\varphi \in \Phi)}{\dfrac{\neg \bigwedge\{\psi \vee \varphi \mid \varphi \in \Phi\}, \psi, \bigwedge \Phi}{\neg \bigwedge\{\psi \vee \varphi \mid \varphi \in \Phi\}, \psi \vee \bigwedge \Phi}\,\texttt{OR}}\,\texttt{AND}$$

$$\dfrac{\cdots\left\{\dfrac{\dfrac{\neg\psi, \psi, \varphi \qquad \dfrac{\neg\varphi, \psi, \varphi}{\neg \bigwedge \Phi, \psi, \varphi}\,\texttt{OR}}{\neg(\psi \vee \bigwedge \Phi), \psi, \varphi}\,\texttt{AND}}{\neg(\psi \vee \bigwedge \Phi), \psi \vee \varphi}\,\texttt{OR}\right\}\cdots\ (\varphi \in \Phi)}{\neg(\psi \vee \bigwedge \Phi), \bigwedge\{\psi \vee \varphi \mid \varphi \in \Phi\}}\,\texttt{AND}$$

$\square$

Notational convention 5.4. For $i < \omega$:

$$\sqrt[i]{\Gamma} := \{\varphi \mid \Box_i \varphi \in \Gamma\}.$$

Lemma 5.5. For every countable set Γ of formulas:

(1) If Γ is $\mathbf{J}$-consistent then, for all $\psi \in \mathcal{E}[\Gamma]$, either $\Gamma \cup \{\psi\}$ or $\Gamma \cup \{\neg\psi\}$ is $\mathbf{J}$-consistent.

(2) If Γ is $\mathbf{J}$-saturated and $\Box_i \psi \in \mathcal{E}[\Gamma] \setminus \Gamma$ then $\sqrt[i]{\Gamma} \cup \{\neg\psi\}$ is $\mathbf{J}$-consistent.

(3) If Γ is finite, then Γ is $\mathbf{J}$-uniform.

(4) If Γ is $\mathbf{J}$-uniform then, for every $\psi \in \mathcal{E}[\Gamma]$, $\Gamma \cup \{\psi\}$ is $\mathbf{J}$-uniform.

Proof. (1) is immediate by the `CUT` rule, and (3) is trivial.
(2): under the assumptions suppose that $\sqrt[i]{\Gamma} \cup \{\neg\psi\}$ is not $\mathbf{J}$-consistent. Then, for some finite set C such that $\Box_i C \subseteq \Gamma$, we have $\mathbf{J} \vdash \neg C, \psi$ and so, by an application of the rule $\mathtt{K}^\sharp$, $\mathbf{J} \vdash \neg\Box_i C, \Box_i\psi$. Thus $\mathbf{J} \rhd \neg\Gamma, \Box_i\psi$, and it now follows by (2) of Lemma 5.2 that $\Box_i\psi \in \Gamma$, against the assumption.
(4): assume that Γ is $\mathbf{J}$-uniform, and suppose $\mathbf{J} \rhd \neg\Gamma, \neg\psi, \varphi$ for all $\varphi \in \Phi$. Then $\mathbf{J} \rhd \neg\Gamma, \bigwedge\{\neg\psi \vee \varphi \mid \varphi \in \Phi\}$ by the $\mathbf{J}$-uniformity of Γ, plus the closure properties of environments $\mathcal{E}[\Gamma]$, in particular (4) of Lemma 2.4. The conclusion follows by `CUT` using the sequent $\neg\bigwedge\{\neg\psi \vee \varphi \mid \varphi \in \Phi\}, \neg\psi, \bigwedge\Phi$ which is derivable in $\mathbf{J}$ by Fact 5.3. $\Box$

Lemma 5.6 (**Saturation**)**.** Every $\mathbf{J}$-*consistent and* $\mathbf{J}$-*uniform* countable set of formulas Γ can be extended to a countable set $\Gamma^* \supseteq \Gamma$ which is $\mathbf{J}$-saturated.

Proof. Assume that Γ is both $\mathbf{J}$-consistent and $\mathbf{J}$-uniform. First of all, let $\chi_0, \chi_1, \chi_2 \ldots$ be an arbitrarily fixed enumeration of the environment $\mathcal{E}[\Gamma]$ of Γ (recall that $|\mathcal{E}[\Gamma]| = \omega$ by (1) of Lemma 2.4).
Define now inductively, for $n \geq 0$, a set $\Gamma_n \subseteq \mathcal{E}[\Gamma]$ provably satisfying:

(1) Γ_n is $\mathbf{J}$-consistent and $\mathbf{J}$-uniform;

(2) $\Gamma \subseteq \Gamma_n \subseteq \Gamma_{n+1}$.

Basis:

- $\Gamma_0 := \Gamma$

Step $\Gamma_k \rightsquigarrow \Gamma_{k+1}$ *:*

Supposing that $\Gamma_0, \ldots, \Gamma_k$ have been defined in a way that (1) and (2) are satisfied, consider the formula χ_k. Since Γ_k is $\mathbf{J}$-consistent, at least one of the sets $\Gamma_k \cup \{\chi_k\}$ and $\Gamma_k \cup \{\neg\chi_k\}$ must be $\mathbf{J}$-consistent by (1) of Lemma 5.5.
— If $\Gamma_k \cup \{\chi_k\}$ is $\mathbf{J}$-consistent, we set:

- $\Gamma_{k+1} := \Gamma_k \cup \{\chi_k\}$.

— If $\Gamma_k \cup \{\chi_k\}$ is not **J**-consistent, and so $\Gamma_k \cup \{\neg\chi_k\}$ is **J**-consistent, we set:
  - $\Gamma_{k+1} := \Gamma_k \cup \{\neg\chi_k\}$,
    in case χ_k is not of the form $\bigwedge \Phi$ for some Φ;
  - $\Gamma_{k+1} := \Gamma_k \cup \{\neg\chi_k, \neg\varphi\}$,
    in case χ_k is of the form $\bigwedge \Phi$, where $\varphi \in \Phi$ is chosen in a way such that $\Gamma_k \cup \{\neg \bigwedge \Phi, \neg\varphi\}$ is **J**-consistent.

Observe that in the latter case such a formula $\varphi \in \Phi$ always exists. Indeed, otherwise we would have

$$\text{for every } \varphi \in \Phi : \ \mathbf{J} \rhd \neg(\Gamma_k \cup \{\neg \bigwedge \Phi\}), \varphi \tag{5.1}$$

and in turn, since $\Gamma_k \cup \{\neg \bigwedge \Phi\}$ is **J**-uniform by (4) of Lemma 5.5 and the fact that Γ_k is **J**-uniform:

$$\begin{aligned} &\mathbf{J} \rhd \neg(\Gamma_k \cup \{\neg \bigwedge \Phi\}), \bigwedge \Phi \\ &\text{i.e.} \quad \mathbf{J} \rhd \neg(\Gamma_k \cup \{\neg \bigwedge \Phi\}) \end{aligned} \tag{5.2}$$

against the **J**-consistency of $\Gamma_k \cup \{\neg\chi_k\}$.

Clearly, (1) and (2) are preserved in the step $\Gamma_k \rightsquigarrow \Gamma_{k+1}$. Finally, set:

$$\Gamma^* := \bigcup_{n \geq 0} \Gamma_n$$

Now $\Gamma^* \supseteq \Gamma$ and Γ^* is easily seen to be saturated. Indeed, it is **J**-consistent by (1) and (2), and $\Gamma^* \cup \neg\Gamma^* = \mathcal{E}[\Gamma] = \mathcal{E}[\Gamma^*]$ by construction. As to the last condition, suppose $\bigwedge \Phi = \chi_n \notin \Gamma^*$. Then, by the construction, $\Gamma_{n+1} = \Gamma_n \cup \{\neg \bigwedge \Phi, \neg\varphi\}$ for some $\varphi \in \Phi$; so $\neg\varphi \in \Gamma^*$ and, by **J**-consistency of Γ^*, $\varphi \notin \Gamma^*$. Hence $\Phi \not\subseteq \Gamma^*$. □

The following Lemma, which makes an essential use of the Barcan Formula, *does hold only* for the calculus $\mathbf{TK}_{\omega_1}$.

Lemma 5.7. Let Γ be a $\mathbf{TK}_{\omega_1}$-saturated set of formulas, and let $\psi \in \mathcal{E}[\Gamma]$ be such that $\Box_i\psi \notin \Gamma$. Then $\sqrt[i]{\Gamma} \cup \{\neg\psi\}$ is $\mathbf{TK}_{\omega_1}$-uniform.

Proof. Under the assumptions, suppose that for each $\varphi \in \Phi$, $\mathbf{TK}_{\omega_1} \rhd \neg(\sqrt[i]{\Gamma} \cup \{\neg\psi\}), \varphi$; that is

$$\text{for each } \varphi \in \Phi : \quad \mathbf{TK}_{\omega_1} \rhd \neg\sqrt[i]{\Gamma}, \psi, \varphi \tag{5.3}$$

Then, by applying `OR` twice, followed by $\mathtt{K}_i$, we have:

$$\text{for each } \varphi \in \Phi : \quad \mathbf{TK}_{\omega_1} \rhd \neg\Box_i\sqrt[i]{\Gamma}, \Box_i(\psi \vee \varphi) \tag{5.4}$$

and, by taking into account that $\Box_i\sqrt[i]{\Gamma} \subseteq \Gamma$:

$$\text{for each } \varphi \in \Phi : \quad \mathbf{TK}_{\omega_1} \rhd \neg\Gamma, \Box_i(\psi \vee \varphi) \tag{5.5}$$

The set Γ is $\mathbf{TK}_{\omega_1}$-saturated by assumption, and by Lemma 2.4 the formulas $\Box_i(\psi \vee \varphi)$ (for $\varphi \in \Phi$) and $\bigwedge\{\Box_i(\psi \vee \varphi) \mid \varphi \in \Phi\}$ all belong to $\mathcal{E}[\Gamma]$. Thus (5.5), together with (2) and (3) of Lemma 5.2, yields:

$$\bigwedge \Box_i\{\psi \vee \varphi \mid \varphi \in \Phi\} \in \Gamma \tag{5.6}$$

On the other side, we have

$$\mathbf{TK}_{\omega_1} \vdash \neg \bigwedge \Box_i\{\psi \vee \varphi \mid \varphi \in \Phi\}, \Box_i(\psi \vee \bigwedge \Phi) \tag{5.7}$$

which is shown as follows by making use of the fact that BF_{ω_1} is derivable in $\mathbf{TK}_{\omega_1}$ (see sect. 3):

$$\dfrac{\dfrac{\dfrac{}{\neg \bigwedge\{\psi \vee \varphi \mid \varphi \in \Phi\}, \psi \vee \bigwedge \Phi}\text{ Fact 5.3}}{\neg\Box_i \bigwedge\{\psi \vee \varphi \mid \varphi \in \Phi\}, \Box_i(\psi \vee \bigwedge \Phi)}\text{ K}}{\neg \bigwedge \Box_i\{\psi \vee \varphi \mid \varphi \in \Phi\}, \Box_i(\psi \vee \bigwedge \Phi)}\text{ CUT with } BF_{\omega_1}$$

Now (5.6) and (5.7) yield

$$\mathbf{TK}_{\omega_1} \rhd \neg\Gamma, \Box_i(\psi \vee \bigwedge \Phi) \tag{5.8}$$

It follows by (2) of Lemma 5.2 that $\Box_i(\psi \vee \bigwedge \Phi) \in \Gamma$, and so that $\psi \vee \bigwedge \Phi \in \sqrt[i]{\Gamma}$. Therefore $\mathbf{TK}_{\omega_1} \rhd \neg\sqrt[i]{\Gamma}, \psi \vee \bigwedge \Phi$ and finally

$$\mathbf{TK}_{\omega_1} \rhd \neg(\sqrt[i]{\Gamma} \cup \{\neg\psi\}), \bigwedge \Phi$$

using a `CUT` with $\mathbf{TK}^{(\sharp)}_{\omega_1} \vdash \neg(\chi_1 \vee \chi_2), \chi_1, \chi_2$. $\square$

We are now ready to define the *canonical models* for $\mathbf{TK}_{\omega_1}$ and $\mathbf{TK}^{\sharp}_{\omega_1}$. Let

- $SAT_{\omega_1} := \{\Gamma \mid \Gamma \subseteq \mathsf{FM}, |\Gamma| \leq \omega, \Gamma \text{ is } \mathbf{TK}_{\omega_1}\text{-saturated}\}$;
- $SAT^\sharp_{\omega_1} := \{\Gamma \mid \Gamma \subseteq \mathsf{FM}, |\Gamma| \leq \omega, \Gamma \text{ is } \mathbf{TK}^\sharp_{\omega_1}\text{-saturated}\}$.

Definition 5.8 ($\mathbf{TK}_{\omega_1}$- and $\mathbf{TK}^\sharp_{\omega_1}$-universal model).

(1) $\mathcal{U}_{\omega_1}$ is the *standard* Kripke model

$$\mathcal{U}_{\omega_1} = \langle SAT_{\omega_1}, \{R_i\}_{i<\omega}, \mathsf{v}\rangle$$

where
- $\Gamma R_i \Delta \;:\Leftrightarrow\; \sqrt[i]{\Gamma} \subseteq \Delta \quad (\Gamma, \Delta \in SAT_{\omega_1}, i < \omega)$;
- $\mathsf{v}(p) := \{\Gamma \in SAT_{\omega_1} \mid p \in \Gamma\}$ for every $p \in \mathsf{Lit}^+$.

(2) $\mathcal{U}^\sharp_{\omega_1}$ is the *generalized* Kripke model

$$\mathcal{U}^\sharp_{\omega_1} = \langle SAT^\sharp_{\omega_1}, \{\mathfrak{R}_i\}_{i<\omega}, \mathsf{v}\rangle$$

where
- for $i < \omega$, $\mathfrak{R}_i := \{R^C_i \mid C \subseteq_{\text{fin}} \mathsf{FM}\}$, with $\Gamma R^C_i \Delta \;:\Leftrightarrow\; \sqrt[i]{\Gamma} \cap C \subseteq \Delta \quad (\Gamma, \Delta \in SAT^\sharp_{\omega_1}, i < \omega)$
- $\mathsf{v}(p) := \{\Gamma \in SAT^\sharp_{\omega_1} \mid p \in \Gamma\}$ for every $p \in \mathsf{Lit}^+$.

Note that, trivially,

$$R^{C\cup D}_i \subseteq R^C_i \cap R^D_i \quad \text{and} \quad R^{C\cup D}_i \in \mathfrak{R}_i,$$

so that $\langle SAT^\sharp_{\omega_1}, \{\mathfrak{R}_i\}_{i<\omega}\rangle$ satisfies the characteristic condition (DD) of a generalized frame.

Theorem 5.9.

(1) For every $\Gamma \in SAT_{\omega_1}$, for every formula $\varphi \in \mathcal{E}[\Gamma]$:

$$\mathcal{U}_{\omega_1}, \Gamma \vDash \varphi \;\Leftrightarrow\; \varphi \in \Gamma$$

(2) For every $\Gamma \in SAT^\sharp_{\omega_1}$, for every formula $\varphi \in \mathcal{E}[\Gamma]$:

$$\mathcal{U}^\sharp_{\omega_1}, \Gamma \vDash_g \varphi \;\Leftrightarrow\; \varphi \in \Gamma$$

Proof. For both (1) and (2) we argue by (transfinite) induction on (the rank of) φ. The cases $\varphi \in \mathsf{Lit}$ and $\varphi \equiv \bigwedge \Phi, \bigvee \Phi$ are immediate by Definition 5.9 and Lemma 5.2, (3) – (4). The case $\varphi \equiv \Box_i \psi$ requires instead separate arguments for $\mathcal{U}_{\omega_1}$ and $\mathcal{U}^\sharp_{\omega_1}$, see below. Finally, the case $\varphi \equiv \widetilde{\Box}_i \psi$ easily reduces to the previous one by (1) of Lemma 5.2.

$(\mathcal{U}_{\omega_1}) : \varphi \equiv \Box_i \psi$.

[$\Leftarrow$]: immediate by the definition of R_i and the induction hypothesis.
[$\Rightarrow$]: suppose that $\Box_i\psi \notin \Gamma$. Then $\Theta := \sqrt[i]{\Gamma} \cup \{\neg\psi\}$ is $\mathbf{TK}_{\omega_1}$-consistent by (2) of Lemma 5.5 as well as $\mathbf{TK}_{\omega_1}$-uniform by Lemma 5.7. Applying Lemma 5.6 to Θ we find a $\Delta \in SAT_{\omega_1}$ such that $\Gamma R_i \Delta$ and $\psi \notin \Delta$. The conclusion $\mathcal{U}_{\omega_1}, \Gamma \nvDash \Box_i\psi$ follows by applying the induction hypothesis.
$(\mathcal{U}^\sharp_{\omega_1}) : \varphi \equiv \Box_i\psi$.
[$\Leftarrow$]: suppose $\Box_i\psi \in \Gamma$, and let $C := \{\psi\}$. Then $R^C_i \in \mathfrak{R}_i$ and, for every $\Delta \in SAT^\sharp_{\omega_1}$ such that $\Gamma R^C_i \Delta$ we obviously have $\psi \in \Delta$, and so also $\mathcal{U}^\sharp_{\omega_1}, \Delta \vDash_g \psi$ by the induction hypothesis. Hence $\mathcal{U}^\sharp_{\omega_1}, \Gamma \vDash_g \Box_i\psi$.
[$\Rightarrow$]: suppose that $\Box_i\psi \notin \Gamma$. To conclude $\mathcal{U}^\sharp_{\omega_1}, \Gamma \nvDash_g \Box_i\psi$ it is sufficient to find, for every finite set C of formulas, a set $\Delta \in SAT^\sharp_{\omega_1}$ such that $\Gamma R^C_i \Delta$ and $\mathcal{U}^\sharp_{\omega_1}, \Delta \nvDash_g \psi$.
This is done as follows: given C, let $D := (\sqrt[i]{\Gamma} \cap C) \cup \{\neg\psi\}$. D is $\mathbf{TK}^\sharp_{\omega_1}$-consistent, for otherwise by the rule $\mathsf{K}^\sharp_i$ we would have $\mathbf{TK}^\sharp_{\omega_1} \rhd \neg\Gamma, \Box_i\psi$ and so by (2) of Lemma 5.2 $\Box_i\psi \in \Gamma$ against the assumption. On the other side, D is finite and so is also $\mathbf{TK}^\sharp_{\omega_1}$-uniform by (3) of Lemma 5.5. It now follows by Lemma 5.6 that there exists a set $\Delta \in SAT^\sharp_{\omega_1}$ such that $D \subseteq \Delta$, and so $\Gamma R^C_i \Delta$, as well as $\neg\psi \in \Delta$, $\psi \notin \Delta$, and finally $\mathcal{U}^\sharp_{\omega_1}, \Delta \nvDash_g \psi$ by the induction hypothesis. $\Box$

Corollary 5.10. *For every finite set $C \subseteq \mathsf{FM}$:*

$$\mathbf{TK}^\sharp_{\omega_1} \nvdash C \quad \Rightarrow \quad \mathcal{U}^\sharp_{\omega_1} \not\models_g C$$

Hence $\mathbf{TK}^\sharp_{\omega_1}$ is sound and complete w.r. to generalized *Kripke semantics.*

Proof. If $\mathbf{TK}^\sharp_{\omega_1} \nvdash C$ then $\neg C$ is clearly $\mathbf{TK}^\sharp_{\omega_1}$-consistent; being a finite set, $\neg C$ is also $\mathbf{TK}^\sharp_{\omega_1}$-uniform by (3) of Lemma 5.5. Then by Lemma 5.6 there exists a set $\Theta \in SAT^\sharp_{\omega_1}$ such that $\neg C \subseteq \Theta$, and so also $C \cap \Theta = \emptyset$. It follows by (2) of Theorem 5.9 that $\mathcal{U}^\sharp_{\omega_1}, \Theta \nvDash_g C$. $\Box$

Corollary 5.11. *For every countable set* $\Gamma \subseteq \mathsf{FM}$*:*

$$\mathbf{TK}_{\omega_1} \nvdash \Gamma \quad \Rightarrow \quad \mathcal{U}_{\omega_1} \not\models \Gamma$$

Hence $\mathbf{TK}_{\omega_1}$ *is sound and complete w.r.t. to* standard *Kripke semantics.*

Proof. If $\mathbf{TK}_{\omega_1} \nvdash \Gamma$ then also $\mathbf{TK}_{\omega_1} \nvdash \bigvee \Gamma$. So we can argue as above taking $C = \{\bigvee \Gamma\}$ and using (1) of Theorem 5.9 to conclude that $\mathcal{U}_{\omega_1}, \Theta \not\models \bigvee \Gamma$, hence also $\mathcal{U}_{\omega_1}, \Theta \not\models \Gamma$, for some $\Theta \in SAT_{\omega_1}$ containing $\bigvee \Gamma$. $\square$

6. $\mathbf{TK}_{\omega_1}$ *does not admit cut-elimination*

As we anticipated, the `CUT` rule cannot be eliminated from $\mathbf{TK}_{\omega_1}$. This will be now demonstrated by exhibiting a suitable example of a valid sequent which is not cut-free derivable in the calculus under investigation.

Below, $\{q_n^k \mid k, n \geq 0\}$ is a set of pairwise distinct positive literals. As previously done, we write '$\vdash_0$' to denote *cut-free* derivability.

Fact 6.1. For every $m \geq 0$ and every $\Phi \subseteq \{q_n^k \mid n \geq 0, k \leq m\}$, if

$$\Theta \subseteq \Delta_\Phi := \neg\Box_i \Phi, \big\{\bigvee_n \neg\Box_i q_n^k \mid k \geq 0\big\}, \Box_i \bigwedge_k q_k^k$$

then $\mathbf{TK}_{\omega_1} \nvdash_0 \Theta$.

Proof. We argue by transfinite induction on the height $\mathtt{h}(\mathfrak{D})$ of $\mathbf{TK}_{\omega_1}$-derivations $\mathfrak{D}$.
Let $\tau < \omega_1$. Assume (I.H.) that for no set Θ satisfying the hypotheses there is a $\mathbf{TK}_{\omega_1}$-derivation $\mathfrak{D} \vdash_0 \Theta$ with $\mathtt{h}(\mathfrak{D}) < \tau$. Suppose, by way of contradiction, that for some $m \geq 0$, some $\Phi \subseteq \{q_n^k \mid n \geq 0, k \leq m\}$ and some $\Lambda \subseteq \Delta_\Phi$ there is a cut-free derivation $\mathfrak{D}$ of Λ with $\mathtt{h}(\Lambda) = \tau$. Let `R` be the final inference of $\mathfrak{D}$. Clearly `R` must be one of `W`, $\mathtt{OR}^+$, $\mathtt{K}_i$.
If $\mathtt{R} = \mathtt{W}$ we are immediately in contradiction with the I.H.
If $\mathtt{R} = \mathtt{OR}^+$, let $\bigvee_n \neg\Box_i q_n^j$ (for some $j \geq 0$) be the principal formula of the inference and $\mathfrak{D}'$ be the subderivation of the premise Λ'. Then $\Lambda' \subseteq \Delta_{\{q_n^k \mid n \geq 0,\, k \leq r\}}$, where $r = \max(m, j)$.

Since $\mathtt{h}(\mathfrak{D}') < \tau$, we are in contradiction with the I.H. again. Finally, if $\mathtt{R} = \mathtt{K}_i$, it is easily seen that from the subderivation of the premise we would also get a derivation

$$\mathfrak{D}' \vdash \neg\Phi', \bigwedge_k q_k^k$$

for some $\Phi' \subseteq \Phi$. But this is clearly impossible by the boundedness condition on Φ and the soundness of $\mathbf{TK}_{\omega_1}$. $\square$

Proposition 6.2. The calculus $\mathbf{TK}_{\omega_1}$ does not admit cut-elimination. For instance, the sequent

$$\Delta \ := \ \big\{\bigvee_n \neg\Box_i q_n^k \mid k \geq 0\big\}, \Box_i \bigwedge_k q_k^k$$

is derivable, but *not cut-free derivable*, in $\mathbf{TK}_{\omega_1}$.

Proof. $\mathbf{TK}_{\omega_1} \nvdash_0 \Delta$ by Fact 6.1, since $\Delta \equiv \Delta_\Phi$ with $\Phi = \emptyset$. On the other side, Δ can be derived as follows by making use of a $\mathtt{CUT}$ with an appropriate instance of BF_{ω_1} (which we know being derivable in $\mathbf{TK}_{\omega_1}$):

$$\dfrac{\dfrac{\cdots\left\{\dfrac{\dfrac{\neg q_k^k, q_k^k}{\neg\Box_i q_k^k, \Box_i q_k^k}\,\mathtt{K}_i}{\bigvee_n \neg\Box_i q_n^k, \Box_i q_k^k}\,\mathtt{OR}\right\}\cdots\ (k\geq 0)}{\{\bigvee_n \neg\Box_i q_n^k \mid k \geq 0\}, \bigwedge_k \Box_i q_k^k}\,\mathtt{W}\,,\mathtt{AND} \qquad \begin{array}{c}\vdots\\ \neg\bigwedge_k \Box_i q_k^k, \Box_i \bigwedge_k q_k^k\end{array}}{\{\bigvee_n \neg\Box_i q_n^k \mid k \geq 0\}, \Box_i \bigwedge_k q_k^k}\,\mathtt{CUT}$$

$\square$

We conclude with a further negative result, showing how a *seemingly natural* way out of the difficulty emerging from Proposition 6.2 is in turn doomed to failure.

Let us consider the calculus $\mathbf{TK}^\circ_{\omega_1}$ obtained from $\mathbf{TK}_{\omega_1}$ by replacing the rule $\mathtt{OR}^+$ with the stronger (and clearly sound) rule

$$\frac{\Gamma, \{\Phi_m\}_{m\in I}}{\Gamma, \{\bigvee \Phi_m\}_{m\in I}}\,\mathtt{OR}^\circ \quad (I \text{ countable})$$

by means of which *countably many* disjunctions can be *simultaneously* introduced.

Indeed, the sequent Δ of Fact 6.2 becomes *cut-free derivable* in $\mathbf{TK}^{\circ}_{\omega_1}$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\ldots \neg q_k^k, q_k^k \ldots \;\; (k \geq 0)}{\{\neg q_k^k \mid k \geq 0\}, \bigwedge_k q_k^k}\;\texttt{W, AND}}{\{\neg\Box_i q_k^k \mid k \geq 0\}, \Box_i \bigwedge_k q_k^k}\;\texttt{K}_i}{\{\neg\Box_i q_n^k \mid k \geq 0, n \geq 0\}, \Box_i \bigwedge_k q_k^k}\;\texttt{W}}{\{\bigvee_n \neg\Box_i q_n^k \mid k \geq 0\}, \Box_i \bigwedge_k q_k^k}\;\texttt{OR}^{\circ}$$

But unfortunately, a new counterexample to cut-elimination comes out! For $n \geq 0$, let

$$\varphi_n := \begin{cases} \neg\Box_i p_0, & \text{if } n = 0; \\ \bigwedge\{\Box_i p_0, \ldots, \Box_i p_k, \neg\Box_i p_{k+1}\}, & \text{if } n = k+1. \end{cases}$$

Fact 6.3. For every $X \subseteq_{\text{fin}} \omega$, if

$$\Theta \subseteq \Gamma_X := \{\varphi_n\}_{n\geq 0}, \{\neg\Box_i p_m\}_{m\in X}, \Box_i \bigwedge_n p_n$$

then $\mathbf{TK}^{\circ}_{\omega_1} \nvdash_0 \Theta$.

Proof. As in the proof of Fact 6.2 we argue by transfinite induction on the height of $\mathbf{TK}^{\circ}_{\omega_1}$-derivations.

Let $\tau < \omega_1$. Assume (I.H.) that for no set Θ satisfying the hypotheses there is a $\mathbf{TK}^{\circ}_{\omega_1}$-derivation $\mathfrak{D} \vdash_0 \Theta$ with $\mathtt{h}(\mathfrak{D}) < \tau$. Suppose, by way of contradiction, that for some set $X \subseteq_{\text{fin}} \omega$ and some $\Lambda \subseteq \Gamma_X$ there is a cut-free derivation $\mathfrak{D}$ of Λ with $\mathtt{h}(\Lambda) = \tau$. Let $\mathtt{R}$ be the final inference of $\mathfrak{D}$. Necessarily $\mathtt{R}$ is be one of $\mathtt{W}$, $\mathtt{AND}$, $\mathtt{K}_i$.

If $\mathtt{R} = \mathtt{W}$ we are immediately in contradiction with the I.H.

If $\mathtt{R} = \mathtt{AND}$, let $\varphi_{j+1} = \bigwedge\{\Box_i p_0, \ldots, \Box_i p_j, \neg\Box_i p_{j+1}\}$ (for some $j \geq 0$) be the principal formula of the inference, and let $\mathfrak{D}'$ be the subderivation of the $j+1$-th premise Λ' (the one having $\neg\Box_i p_{j+1}$ as secondary formula) of this inference. Then $\Lambda' \subseteq \Gamma_{X\cup\{j+1\}}$, and since $\mathtt{h}(\mathfrak{D}') < \tau$ we are in contradiction with the I.H. again.

If $\mathtt{R} = \mathtt{K}_i$, then there would be a finite set $Y = X \cup \{0\} \subseteq \omega$

and a derivation

$$\mathfrak{D}' \vdash \{\neg p_k\}_{k \in Y}, \bigwedge_n p_n$$

which is clearly impossible by the finiteness of Y and the soundness of $\mathbf{TK}^{\circ}_{\omega_1}$. $\square$

Proposition 6.4. The calculus $\mathbf{TK}^{\circ}_{\omega_1}$ does not admit cut-elimination. For instance, the sequent

$$\Gamma \ := \ \{\varphi_n\}_{n \geq 0}, \Box_i \bigwedge_n p_n$$

is derivable, but *not cut-free derivable*, in $\mathbf{TK}^{\circ}_{\omega_1}$.

Proof. $\mathbf{TK}^{\circ}_{\omega_1} \nvdash_0 \Gamma$ by Fact 6.3, since $\Gamma \equiv \Gamma_X$ with $X = \emptyset$. On the other side, Γ can be derived (in fact, already in $\mathbf{TK}_{\omega_1}$) by making use of `CUT`. First of all, we verify:

$$\text{(6.1)} \qquad \mathbf{TK}_{\omega_1} \vdash_0 \varphi_0, \ldots, \varphi_m, \Box_i p_m \quad \text{for each } m \geq 0.$$

This is easily proved by induction on m:

— $m = 0$:

$$\frac{\neg p_0, p_0}{\neg\Box_i p_0, \Box_i p_0}\ \mathtt{K}_i$$

— $m = k + 1$:

$$\frac{\overset{\text{I.H.}}{\varphi_0, \Box_i p_0} \quad \cdots \quad \overset{\text{I.H.}}{\varphi_0, \ldots, \varphi_k, \Box_i p_k} \quad \dfrac{\neg p_{k+1}, p_{k+1}}{\neg\Box_i p_{k+1}, \Box_i p_{k+1}}\ \mathtt{K}_i}{\varphi_0, \ldots, \varphi_k, \bigwedge\{\Box_i p_0, \ldots, \Box_i p_k, \neg\Box_i p_{k+1}\}, \Box_i p_{k+1}}\ \mathtt{W,\ AND}$$

Next, using BF_{ω_1}, we obtain the following derivation of Γ in $\mathbf{TK}_{\omega_1}$:

$$\frac{\dfrac{\cdots\left\{\begin{array}{c}(6.1)\\ \varphi_0, \ldots, \varphi_m, \Box_i p_m\end{array}\right\}\cdots\ (m \geq 0)}{\{\varphi_n\}_{n \geq 0}, \bigwedge_n \Box_i p_n}\ \mathtt{W\ ,\ AND} \qquad \begin{array}{c}\vdots\\ \neg\bigwedge_n \Box_i p_n, \Box_i \bigwedge_n p_n\end{array}}{\{\varphi_n\}_{n \geq 0}, \Box_i \bigwedge_n p_n}\ \mathtt{CUT}$$

$\square$

References

[1] R. Fagin, J. Y. Halpern, Y. Moses and M. Y. Vardi, *Reasoning About Knowledge*, The MIT Press, Cambridge 1995.

[2] S. Feferman, *Lectures on proof theory*, in M. H. Löb (ed.), *Proceedings of the Summer School in Logic, Leeds 1967*, Lecture Notes in Mathematics 70, Springer, Berlin 1967, pp. 1 – 107.

[3] E. G. K. Lopez-Escobar, *An interpolation theorem for denumerably long formulas*, «Fundamenta Mathematicæ», 57, 1965, pp. 253 – 272.

[4] E. G. K. Lopez-Escobar, *Remarks on an Infinitary Language with Constructive Formulas*, «The Journal of Symbolic Logic», 32, 1967, pp. 305 – 318.

[5] M. Ohnishi and K. Matsumoto, *Gentzen method in modal calculi*, «Osaka Mathematical Journal», 9, (1957), pp. 113 - 130.

[6] S. Radev, *Infinitary propositional normal modal logic*, «Studia Logica», 46, 1987, pp. 291 – 309.

[7] W. W. Tait, *Normal derivability in classical logic*, in J. Barwise (ed.), *The Syntax and Semantics of Infinitary Languages*, Lecture Notes in Mathematics 72, Springer, Berlin 1968, pp. 204 – 236.

[8] Y. Tanaka, *Kripke Completeness of Infinitary Predicate Multimodal Logics*, «Notre Dame Journal of Formal Logic», 40, 1999, pp. 327 – 339.

[9] Y. Tanaka, *Cut-elimination Theorems for Some Infinitary Modal Logics*, «Mathematical Logic Quarterly», 47, 2001, pp. 326 – 340.

[10] Y. Tanaka and H. Ono, *Rasiowa-Sikorski Lemma and Kripke completeness of predicate and infinitary modal logics*, in M. Zakharyaschev, K. Segerberg, M. de Rijke and H. Wansing (eds.), *Advances in Modal Logic 98*, Vol. 2, CSLI Publications, Stanford 2000, pp. 419 – 437.